Synthesis of Finite-Interval H_∞ Controllers by State-Space Methods

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In this paper, a state-space formulation of the H_∞ optimal-control problem is given. Assuming a finite interval of control, the problem of synthesizing a finite-interval H_∞ controller is converted into an optimization problem in which a parameter occurring in a boundary-value problem needs to be maximized. An optimality condition for the maximization of this parameter is given. The proposed method makes use of the observer-based parameterization of all stabilizing controllers. An example is given.

I. Introduction

THE H_∞ optimal-control theory has been pioneered by Zames,[1] and important contributions have been made by Francis and Doyle.[2,3] Recent work[4] indicates that the theory has important applications in the design of flight control systems when combined with the so-called μ-analysis.[5,6] For a good account of the μ-analysis and the associated numerical algorithms, see Ref. 7.

In this paper, a variant of the H_∞ control problem is considered in terms of state-space formulation. Optimization routines are needed for the synthesis of the final controller. The formulation is new, and the approach is based on considering optimal control problems with finite terminal time in which the cost functional is a quotient of two definite integrals. The mathematical theory behind the method is given in Refs. 8 and 9. Additional related material can be found in Refs. 10–13.

Other authors have considered the H_∞ problem from different points of view. In Ref. 14, a parameterization of all stabilizing controllers that achieve a specified H_∞ norm bound is given in a specialized case. The computation of the controller involves the solution of two Riccati equations. This result has recently been extended to the general case.[15] In Ref. 16, the H_∞ problem is solved by introducing a generalized algebraic operation called conjugation. The approach again yields two Riccati equations whose solution leads to the synthesis of a controller. Reference 17 solves a certain linear-quadratic-Gaussian problem with a side constraint on the H_∞ norm of the closed-loop transfer function. In this approach it is necessary to solve three coupled Riccati equations. In special cases these three equations can be reduced to two Riccati equations.

Our approach is fundamentally different and it results in a two-point boundary-value problem. We assume a finite interval of control. The approach has the advantage of being applicable to time-varying systems with observer-based controllers and dynamic controllers. Reference 18 contains one such application in which the objective is to maximize the disturbance rejection capacity of a time-varying linear system. Also, given a controller, it is important to know the performance measure of the controller. For the general time-varying system with a given controller, the parameter λ of Sec. III gives a measure of the performance of the controller.

Our time-domain approach has several advantages even in the case of time-invariant systems. First of all, it provides an alternate new approach for the computation of finite-interval H_∞ controllers. Also in our formulation it is possible to consider ratios of weighted error energy and weighted exogenous input energy. The H_∞ optimization algorithms usually cannot handle time domain specifications. In our optimization algorithm, it is possible to include time-domain constraints. Also, the time-domain approach is convenient for handling parameter uncertainties. Our current investigations indicate that our numerical optimization approach has considerable promise in the solution of the important robust performance problem, viz., how to achieve maximum performance and required robustness under parameter variations.

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II. State-Space Formulation of the H_∞ Problem

The standard H_∞ problem can be stated with reference to Fig. 1. In Fig. 1, w, u, z, and y denote the exogenous input (command signals, disturbances, sensor noises, etc.), the control input, the output to be controlled, and the measured output, respectively. The plant $G(s)$ and the controller $K(s)$ are assumed to be real-rational and proper. Partition G as

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \tag{1}$$

The equations corresponding to Fig. 1 are

$$z = G_{11}w + G_{12}u, \qquad y = G_{21}w + G_{22}u, \qquad u = Ky \tag{2}$$

The standard H_∞ problem is to find a real-rational proper K that minimizes the H_∞ norm of the transfer matrix from w to z under the constraint that K stabilize G.

In terms of state space-equations, $G(s)$ is written as

$$\dot{x} = Ax + B_1w + B_2u \tag{3a}$$

$$z = C_1x + D_{11}w + D_{12}u \tag{3b}$$

$$y = C_2x + D_{21}w + D_{22}u \tag{3c}$$

Doyle[6] showed that every stabilization procedure can be realized as an observer-based controller by adding stable dynamics to the plant. The realization of the observer-based controller is shown in Fig. 2 where the stable dynamics added is represented by $Q(s)$, with $Q(s)$ proper and $I - D_{22}Q(\infty)$ invertible.

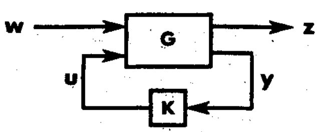

Fig. 1 Standard block diagram.

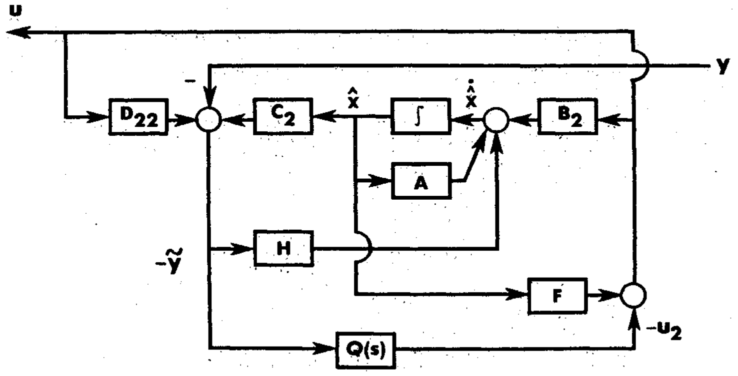

Fig. 2 Observer-based controller parametrization.

In Fig. 2, F and H are chosen such that $A + B_2F$ and $A + HC_2$ are stable. Assume that $Q(s)$ is described by the minimal representation

$$\dot{q} = \tilde{A}q + \tilde{B}\tilde{y}, \qquad u_2 = \tilde{C}q + \tilde{D}\tilde{y} \tag{4}$$

Following the notation of Ref. 19, define the following quantities:

$$\beta_1 = -H - (B_2 + HD_{22})(I - \tilde{D}D_{22})^{-1}\tilde{D} \tag{5a}$$

$$\beta_2 = \tilde{B} + \tilde{B}D_{22}(I - \tilde{D}D_{22})^{-1}\tilde{D} \tag{5b}$$

$$\gamma_1 = F + (I - \tilde{D}D_{22})^{-1}\tilde{D}(C_2 + D_{22}F) \tag{5c}$$

$$\gamma_2 = -(I - \tilde{D}D_{22})^{-1}\tilde{C} \tag{5d}$$

$$\alpha_{11} = A + HC_2 + (B_2 + HD_{22})\gamma_1$$

$$= A + B_2F - \beta_1(C_2 + D_{22}F) \tag{5e}$$

$$\alpha_{12} = (B_2 + HD_{22})\gamma_2 \tag{5f}$$

$$\alpha_{21} = -\beta_2(C_2 + D_{22}F) \tag{5g}$$

$$\alpha_{22} = \tilde{A} - \tilde{B}D_{22}\gamma_2 \tag{5h}$$

$$\kappa = -(I - \tilde{D}D_{22})^{-1}\tilde{D} \tag{5i}$$

Then the closed-loop system is given by

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathcal{A}_{13} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \mathcal{A}_{23} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ q \end{bmatrix} + \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \\ \mathcal{B}_3 \end{bmatrix} w \tag{6}$$

$$y = (I - D_{22}\kappa)^{-1}[C_2x + D_{22}\gamma_1\hat{x} + D_{22}\gamma_2q + D_{21}w] \tag{7}$$

$$z = C_1x + D_{11}w + D_{12}(\gamma_1\hat{x} + \gamma_2q + \kappa y) \tag{8}$$

where

$$\mathcal{A}_{11} = A + B_2\kappa(I - D_{22}\kappa)^{-1}C_2 \tag{9a}$$

$$\mathcal{A}_{12} = B_2\gamma_1 + B_2\kappa(I - D_{22}\kappa)^{-1}D_{22}\gamma_1 \tag{9b}$$

$$\mathcal{A}_{13} = B_2\gamma_2 + B_2\kappa(I - D_{22}\kappa)^{-1}D_{22}\gamma_2 \tag{9c}$$

$$\mathcal{A}_{21} = \beta_1(I - D_{22}\kappa)^{-1}C_2 \tag{9d}$$

$$\mathcal{A}_{22} = \alpha_{11} + \beta_1(I - D_{22}\kappa)^{-1}D_{22}\gamma_1 \tag{9e}$$

$$\mathcal{A}_{23} = \alpha_{12} + \beta_1(I - D_{22}\kappa)^{-1}D_{22}\gamma_2 \tag{9f}$$

$$\mathcal{A}_{31} = \beta_2(I - D_{22}\kappa)^{-1}C_2 \tag{9g}$$

$$\mathcal{A}_{32} = \alpha_{21} + \beta_2(I - D_{22}\kappa)^{-1}D_{22}\gamma_1 \tag{9h}$$

$$\mathcal{A}_{33} = \alpha_{22} + \beta_2(I - D_{22}\kappa)^{-1}D_{22}\gamma_2 \tag{9i}$$

$$\mathcal{B}_1 = B_1 - B_2\tilde{D}(I - D_{22}\kappa)^{-1}D_{21} \tag{9j}$$

$$\mathcal{B}_2 = -(H + B_2\tilde{D})D_{21} \tag{9k}$$

$$\mathcal{B}_3 = \tilde{B}D_{21} \tag{9l}$$

Consider Eqs. (4-9). Now the H_∞ control problem is to find among all sets of matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ that give a stable transfer matrix from $\tilde{y}$ to u_2 (see Fig. 2), one for which the H_∞ norm of the transfer matrix from w to z is minimized.

The preceding problem is equivalent to the following problem. Suppose $\tilde{A}$ is selected to be a stable matrix. For fixed $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$, let

$$\lambda = \inf_{w} \frac{\int_0^\infty w^*(t)w(t)\,\mathrm{d}t}{\int_0^\infty z^*(t)z(t)\,\mathrm{d}t} \tag{10}$$

where * denotes the matrix transpose. Now find the values of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ that make λ a maximum. The initial conditions for the variables x, $\hat{x}$, and q are of course zero.

It is clear that the H_∞ norm of the transfer function from w to z is $1/\sqrt{\lambda}$, and the objective is to minimize the H_∞ norm by choosing a controller. However, since the basic theory for cost functionals of the form of a quotient of definite integrals is given in Refs. 8-13, we follow the same notation as in these references and consider the equivalent problem of maximizing λ.

The input $w(t)$ considered in the preceding problem is an element of $L_2(0,\infty)$. However, in many physical systems, the control interval is finite. For example, in the case of an advanced fighter, most maneuvers are accomplished in the course of a few seconds. Thus, in the next section, we consider an approximate H_∞ problem in the sense that the control interval will be finite. If the integration limit T in, say Eq. (13), approaches infinity, then $\sqrt{\lambda}$ is the inverse of the H_∞ norm of the transfer matrix from w to z. For lack of a better term, we call this a finite-time H_∞ problem. On the other hand, the problem will be more general in the sense that time-varying linear systems and a broader class of performance indices will be considered in Sec. III.

To motivate the problem considered in the next section, let $\mathcal{x} = (x^*, \hat{x}^*, q^*)^*$. Equations (6-8) are written as

$$\dot{\mathcal{x}} = \mathcal{A}\mathcal{x} + \mathcal{B}\mathcal{w}, \qquad \mathcal{x}(0) = 0, \qquad \mathcal{w} = w \tag{11}$$

$$\mathcal{z} = z = \mathcal{C}\mathcal{x} + \mathcal{D}\mathcal{w} \tag{12}$$

where the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ depend on $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$. Let the control interval be $[0, T]$. For fixed $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and

$\tilde{D}$, with $\tilde{A}$ being stable, let

$$\lambda = \inf_{w} \frac{\int_0^T w^*(t)w(t)\,dt}{\int_0^T z^*(t)z(t)\,dt} \tag{13}$$

Using an optimization routine, find the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ for which λ is maximized.

III. Optimality Conditions

In this section we develop conditions for determining λ in a general case, which subsumes the problem considered at the end of Sec. II. These conditions will be developed for time-varying systems. The system equations are given by

$$\dot{x} = \mathcal{A}(t)x + \mathcal{B}(t)w, \qquad x(t_0) = 0 \tag{14}$$

The problem on hand is to select w that minimizes the performance index given by

$$J(w) = \frac{\int_{t_0}^T \left\{ \tfrac{1}{2} x^*\mathcal{R}_1 x + x^*\mathcal{R}_2 w + \tfrac{1}{2} w^*\mathcal{R}_3 w \right\} dt}{\int_{t_0}^T \left\{ \tfrac{1}{2} x^*\mathcal{W}_1 x + x^*\mathcal{W}_2 w + \tfrac{1}{2} w^*\mathcal{W}_3 w \right\} dt} \tag{15}$$

Note that the performance index given by Eq. (13) can be regarded as a special case of Eq. (15) since $z = \mathcal{C}x + \mathcal{D}w$. To get the performance index of Eq. (13), set $\mathcal{R}_1 = \mathcal{R}_2 = 0$, $\mathcal{W}_1 = \mathcal{C}^*\mathcal{C}$, $\mathcal{W}_2 = \mathcal{C}^*\mathcal{D}$, and $\mathcal{W}_3 = \mathcal{D}^*\mathcal{D}$ in Eq. (15). In Eq. (15), we assume that the weighting matrices $\mathcal{R}_1$, $\mathcal{R}_3$, $\mathcal{W}_1$, and $\mathcal{W}_3$ are symmetric, and the integrands of both the numerator and denominator are non-negative for each $w(t)$. Further, we assume that there is some $w(t)$ for which the denominator is positive. Let $\lambda = \inf_{w} J(w)$. We also assume that $\mathcal{R}_3 - \lambda\mathcal{W}_3$ is nonsingular.

Cost functionals of the form of Eq. (15) have been considered, for example, in Refs. 8 and 9 in a different context. Here we treat a problem in which the performance index consists of quadratic terms. This results in a tractable boundary value problem.

Since the infimum of Eq. (15) is λ, we have

$$\int_{t_0}^T \left\{ \tfrac{1}{2} x^*\mathcal{R}_1 x + x^*\mathcal{R}_2 w + \tfrac{1}{2} w^*\mathcal{R}_3 w \right\} dt - \lambda \int_{t_0}^T \left\{ \tfrac{1}{2} x^*\mathcal{W}_1 x + x^*\mathcal{W}_2 w + \tfrac{1}{2} w^*\mathcal{W}_3 w \right\} dt \geq 0 \tag{16}$$

for all pairs (w, x) that satisfy Eq. (14). Thus, if w minimizes the cost functional in Eq. (15), it also minimizes the alternate cost functional

$$J_1(w) = \int_{t_0}^T \left\{ \tfrac{1}{2} x^*(\mathcal{R}_1 - \lambda\mathcal{W}_1)x + x^*(\mathcal{R}_2 - \lambda\mathcal{W}_2)w + \tfrac{1}{2} w^*(\mathcal{R}_3 - \lambda\mathcal{W}_3)w \right\} dt \tag{17}$$

The necessary conditions for optimal $w(t)$ can be stated as follows.

Theorem 1: Consider the system given by Eq. (14) with the performance index given by Eq. (15). If $w(t)$ minimizes Eq. (15), then there exists an adjoint vector $\psi(t)$, not identically zero, such that

$$\frac{d\psi}{dt} = -\mathcal{A}^*\psi + (\mathcal{R}_1 - \lambda\mathcal{W}_1)x + (\mathcal{R}_2 - \lambda\mathcal{W}_2)w, \qquad \psi(T) = 0 \tag{18}$$

$$w(t) = (\mathcal{R}_3 - \lambda\mathcal{W}_3)^{-1}\left\{ \mathcal{B}^*\psi - (\mathcal{R}_2 - \lambda\mathcal{W}_2)^* x \right\} \tag{19}$$

Proof: Consider the alternate cost functional given by Eq. (17). By the maximal principle,[20] the Hamiltonian is given by

$$\mathcal{H}(\psi, x, w) = \psi^*(\mathcal{A}x + \mathcal{B}w) - \left\{ \tfrac{1}{2} x^*(\mathcal{R}_1 - \lambda\mathcal{W}_1)x + x^*(\mathcal{R}_2 - \lambda\mathcal{W}_2)w + \tfrac{1}{2} w^*(\mathcal{R}_3 - \lambda\mathcal{W}_3)w \right\} \tag{20}$$

The adjoint vector $\psi(t)$ satisfies

$$\frac{d\psi}{dt} = -\frac{\partial\mathcal{H}}{\partial x} \tag{21}$$

with the transversality condition $\psi(T) = 0$. Equation (18) is obtained from Eq. (21). Optimal $w(t)$ is obtained by setting $\partial\mathcal{H}/\partial w = 0$ and is given by Eq. (19).

Let $\mathcal{V}_i = \mathcal{R}_i - \lambda\mathcal{W}_i$ for $i = 1,2,3$. We have a two-point boundary-value problem given by

$$\begin{bmatrix} \dot{x} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \mathcal{A} - \mathcal{B}\mathcal{V}_3^{-1}\mathcal{V}_2^* & \mathcal{B}\mathcal{V}_3^{-1}\mathcal{B}^* \\ \mathcal{V}_1 - \mathcal{V}_2\mathcal{V}_3^{-1}\mathcal{V}_2^* & -\mathcal{A}^* + \mathcal{V}_2\mathcal{V}_3^{-1}\mathcal{B}^* \end{bmatrix} \begin{bmatrix} x \\ \psi \end{bmatrix} \tag{22}$$

with

$$x(t_0) = 0, \qquad \psi(T) = 0 \tag{23}$$

We now show that the minimum value of Eq. (15) is the least positive λ for which Eqs. (22) and (23) have a solution, with

$$\int_{t_0}^T \left\{ \tfrac{1}{2} x^*\mathcal{W}_1 x + x^*\mathrm{W}_2 w + \tfrac{1}{2} w^*\mathcal{W}_3 w \right\} dt > 0$$

Theorem 2: Consider the boundary-value problem given by Eqs. (22) and (23). Let λ be the least-positive value for which the boundary-value problem has a solution with

$$\int_{t_0}^T \left\{ \tfrac{1}{2} x^*\mathcal{W}_1 x + x^*\mathcal{W}_2 w + \tfrac{1}{2} w^*\mathcal{W}_3 w \right\} dt > 0$$

Define $w(t) = \mathcal{V}_3^{-1}\{\mathcal{B}^*\psi - \mathcal{V}_2^* x\}$. Then, λ is the minimum value of Eq. (15), and w is an optimal input.

Proof: From Theorem 1, it follows that if $w(t)$ is optimal, then the boundary-value problem of Eqs. (22) and (23) is satisfied for the optimal value of λ. Now suppose the boundary-value problem is satisfied for some λ such that the corresponding solution (x, ψ) gives the denominator of Eq. (15) a positive value [with $w(t) \triangleq \mathcal{V}_3^{-1}\{\mathcal{B}^*\psi - \mathcal{V}_2^* x\}$]. We show that the performance index corresponding to (x, ψ) is λ.

Let $(.,.)$ denote the standard inner product in a real Euclidean space. We have

$$(\mathcal{R}_3 - \lambda\mathcal{W}_3)w = \mathcal{B}^*\psi - (\mathcal{R}_2 - \lambda\mathcal{W}_2)^* x \tag{24}$$

Thus

$$\int_{t_0}^T \left\{ (w, \mathcal{R}_3 w) - \lambda(w, \mathcal{W}_3 w) \right\} dt = \int_{t_0}^T \left\{ (w, \mathcal{B}^*\psi) - (w, \mathcal{R}_2^* x) + \lambda(w, \mathcal{W}_2^* x) \right\} dt \tag{25}$$

Since $\mathcal{B}w = \dot{x} - \mathcal{A}x$, the first integral on the right side of Eq. (25) is

$$\int_{t_0}^T (w, \mathcal{B}^*\psi)\,dt = \int_{t_0}^T \left\{ (\dot{x}, \psi) - (\mathcal{A}x, \psi) \right\} dt \tag{26}$$

After integrating the right side of Eq. (26) by parts and using $x(t_0) = \psi(T) = 0$,

$$\int_{t_0}^T (w, \mathcal{B}^*\psi)\,dt = \int_{t_0}^T \left\{ (x, \mathcal{R}_1 x) - (x, \mathcal{R}_2 w) + \lambda(x, \mathcal{W}_1 x) + \lambda(x, \mathcal{W}_2 w) \right\} dt \tag{27}$$

Combining Eqs. (25) and (27), we get

$$\int_{t_0}^{T}\Big\{(x,\mathcal{R}_1 x)+2(x,\mathcal{R}_2 w)+(w,\mathcal{R}_3 w)\Big\}\,dt=\lambda\int_{t_0}^{T}\Big\{(x,\mathcal{W}_1 x)+2(x,\mathcal{W}_2 w)+(w,\mathcal{W}_3 w)\Big\}\,dt \tag{28}$$

Thus λ is the cost associated with (x,ψ). Therefore if λ is the least-positive value for which the boundary value problem [Eqs. (22) and (23)] has a solution (x,ψ) with the corresponding denominator of Eq. (15) being positive, then x must be an optimal trajectory.

If the system and weighting matrices are functions of a finite number of parameters, these parameters can be varied to maximize λ. In Sec. II, since the system matrices and the weighting matrices depend on $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$, an optimization routine needs to be employed with respect to these quantities to maximize λ.

IV. Optimality Conditions for the Maximization of λ

We consider again the standard time-invariant H_∞ problem. In this section we derive a condition that needs to be satisfied when λ is maximized. For this, consider Eqs. (14) and (15). Note that for the standard H_∞ problem of Sec. II, the system and weighting matrices depend on $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$. These constitute the set of independent variables. The variations in the system and weighting matrices can be explicitly expressed in terms of variations in $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$. However the optimality conditions are extremely complicated to derive in such a case. The derivation can be simplified a little by assuming that $D_{22} = 0$ [see Eq. (3)]. However, we only attempt to derive the basic optimality condition here.

Consider Eqs. (22) and (23). Let $\hat{A} = \mathcal{A} - BV_3^{-1}V_2^*$, $\hat{B} = \mathcal{B}\mathcal{V}_3^{-1}\mathcal{B}^*$, and $\hat{C} = \mathcal{V}_1 - \mathcal{V}_2\mathcal{V}_3^{-1}\mathcal{V}_2^*$. Suppose $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ maximize λ. Let $\delta\tilde{A}$, $\delta\tilde{B}$, $\delta\tilde{C}$, and $\delta\tilde{D}$ denote elemental perturbations in $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$, respectively. Also, denote the corresponding perturbations in $\hat{A}$, $\hat{B}$, $\hat{C}$, x, ψ, and λ by $\delta\hat{A}$, $\delta\hat{B}$, $\delta\hat{C}$, x_1, ψ_1, and μ, respectively. Note that if λ is a maximum, $\mu = 0$. Thus, we have the following set of equations:

$$\dot{x} = \hat{A}x + \hat{B}\psi \tag{29}$$

$$\dot{\psi} = \hat{C}x - \hat{A}^*\psi \tag{30}$$

$$x(t_0) = \psi(T) = 0 \tag{31}$$

$$\dot{x}_1 = \hat{A}x_1 + \hat{B}\psi_1 + \delta\hat{A}x + \delta\hat{B}\psi \tag{32}$$

$$\dot{\psi}_1 = \hat{C}x_1 - \hat{A}^*\psi_1 + \delta\hat{C}x - \delta\hat{A}^*\psi \tag{33}$$

$$x_1(t_0) = \psi_1(T) = 0 \tag{34}$$

From Eq. (33), we have

$$\int_{t_0}^{T} x^*\dot{\psi}_1\,dt = \int_{t_0}^{T}\Big\{x^*\hat{C}x_1 - x^*\hat{A}^*\psi_1 + x^*\delta\hat{C}x - x^*\delta\hat{A}\psi\Big\}\,dt \tag{35}$$

Also, by an integration by parts

$$\int_{t_0}^{T} x^*\dot{\psi}_1\,dt = -\int_{t_0}^{T}\Big\{x^*\hat{A}^*\psi_1 + \psi^*\hat{B}\psi_1\Big\}\,dt \tag{36}$$

From Eqs. (35) and (36),

$$-\int_{t_0}^{T}\psi^*\hat{B}\psi_1\,dt = \int_{t_0}^{T}\Big\{x^*\hat{C}x_1 + x^*\delta\hat{C}x - x^*\delta\hat{A}\psi\Big\}\,dt \tag{37}$$

From Eq. (30)

$$\hat{C}x = \dot{\psi} + \hat{A}^*\psi \tag{38}$$

Note that $\hat{C}^* = \hat{C}$. Substituting Eq. (38) in Eq. (37) and integrating by parts, we get

$$2\int_{t_0}^{T} x^*\delta\hat{A}\psi\,dt + \int_{t_0}^{T}\psi^*\delta\hat{B}\psi\,dt - \int_{t_0}^{T} x^*\delta\hat{C}x\,dt = 0 \tag{39}$$

The above equation needs to be satisfied for all elemental perturbations in $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$.

V. Numerical Example

As an example, we consider the tracking problem given in Ref. 2. The plant is given by

$$P(s) = \frac{s-1}{s(s-2)} \tag{40}$$

The tracking error signal is $r - v$. The weighting filter $W(s)$ in Fig. 3 is given by

$$W(s) = \frac{s+1}{10s+1} \tag{41}$$

The objective in Ref. 2 was to choose $K_1(s)$ and $K_2(s)$ such that the H_∞ norm of the transfer function from w to v is minimized. Our objective in this section is to synthesize u using the theory of this paper such that the minimum of

$$\frac{\int_0^{10} w^2(t)\,dt}{\int_0^{10}\Big\{(r-v)^2 + u^2\Big\}\,dt} \tag{42}$$

is maximized.

Converting the plant equations to state-space form, we have

$$\dot{x}_1 = -0.1x_1 + w \tag{43a}$$

$$\dot{x}_2 = u \tag{43b}$$

$$\dot{x}_3 = 2x_3 + u \tag{43c}$$

$$r = 0.1w + 0.09x_1 \tag{43d}$$

$$v = 0.5x_2 + 0.5x_3 \tag{43e}$$

The matrices corresponding to Eq. (3) are given by

$$A = \begin{bmatrix} -0.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0.09 & -0.5 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.09 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

$$D_{11} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}, \quad D_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The matrices F and H are chosen such that $A + B_2F$ and $A + HC_2$ are stable. The choice is the same as that in Ref. 2 and is given by

$$F = [0 \;\; 0.5 \;\; -4.5], \qquad H = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -9 \end{bmatrix}$$

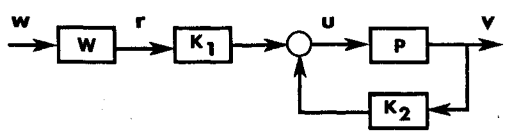

Fig. 3 Block diagram for the tracking problem.

Assume that $Q(s)$ is described by the three-dimensional system

$$\dot{q} = \tilde{A}q + \tilde{B}\tilde{y} \tag{44a}$$

$$u_2 = \tilde{C}q + \tilde{D}\tilde{y} \tag{44b}$$

Let $x = (x_1\ x_2\ x_3)^*$. Then the state equations for the approximate H_∞ problem become

$$\dot{x} = (A - B_2\tilde{D}C_2)x + B_2(F + \tilde{D}C_2)\hat{x} - B_2\tilde{C}q + (B_1 - B_2\tilde{D}D_{21})w \tag{45}$$

$$\dot{\hat{x}} = (A + HC_2 + B_2F + B_2\tilde{D}C_2)\hat{x} - (HC_2 + B_2\tilde{D}C_2)x - B_2\tilde{C}q - (HD_{21} + B_2\tilde{D}D_{21})w \tag{46}$$

$$\dot{q} = \tilde{A}q + \tilde{B}C_2(x - \hat{x}) + \tilde{B}D_{21}w \tag{47}$$

with the initial conditions being zero. The performance index is

$$\left[\int_0^{10} w^2(t)\,\mathrm{d}t\right] \Big/ \left[\int_0^{10}\left\{(0.1w + 0.09x_1 - 0.5x_2 - 0.5x_3)^2 + [F\hat{x} - (\tilde{C}q + \tilde{D}C_2x - \tilde{D}C_2\hat{x} + \tilde{D}D_{21}w)]^2\right\}\mathrm{d}t\right] \tag{48}$$

Assuming values for $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, we can find λ using the theory given in Sec. III. Let

$$\phi = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

be the transition matrix corresponding to Eq. (22). Satisfaction of Eq. (23) gives rise to the condition that det $[\phi_{22}(10)] = 0$. Thus λ is found by making use of a sign change of det $[\phi_{22}(10)]$ over a range of values of λ. As an alternative, it is possible to use λ as an additional state variable and to employ quasilinearization to determine the first positive value of λ that allows satisfaction of the boundary conditions.[21,22] In our numerical experiments, much of the computer execution time was consumed by the calculation of λ for a given controller. Efforts are underway to make the computation of λ more efficient. The technique of quasilinearization just discussed is being applied toward this goal.

The transition matrix $\phi(10)$ was found using the following formula.[23]

Let $h = 10/2^8$. Represent the system matrix in Eq. (22) by $\mathfrak{M}$. Then

$$\phi(10) = \left\{\left[I - \frac{1}{2}h\mathfrak{M} + \frac{1}{12}h^2\mathfrak{M}^2\right]^{-1}\left[I + \frac{1}{2}h\mathfrak{M} + \frac{1}{12}h^2\mathfrak{M}^2\right]\right\}^{2^8} \tag{49}$$

Using the preceding procedure, we can iterate on $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ to maximize λ. Note that once $\phi(h)$ is calculated, only eight matrix multiplications are needed to evaluate $\{\phi(h)\}^{2^8}$.

Initially the following values were assumed for the control matrices:

$$\tilde{A} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\tilde{C} = [1 \quad 1 \quad 1], \qquad \tilde{D} = [1 \quad 1]$$

Using the Rosenbrock hill-climbing algorithm,[24] the elements of the matrices were varied to maximize λ. The algorithm usually leads to only local maxima. Note that $Q(s)$ is stable if and only if $\tilde{A}$ is stable. This was not introduced as a constraint in the optimization algorithm since the unconstrained run yielded a stable $\tilde{A}$. The Fortran program was run on a Zenith Z-248 personal computer in double precision using the Microsoft Optimizing Compiler Version 4.01. A local maximum of $\lambda = 14.8$ was obtained for the following values of A, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$:

$$\tilde{A} = \begin{bmatrix} -2.04 & 0.318 & 0.023 \\ -0.026 & -1.632 & -0.028 \\ -0.052 & 0.358 & -2.054 \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} 0.945 & 9.973 \\ -1.046 & 33.028 \\ 0.946 & -1.056 \end{bmatrix}$$

$$\tilde{C} = [0.944 \quad 1.48 \quad 1.018], \qquad \tilde{D} = [0.986 \quad 41.92]$$

After several runs with various initial values for $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$, the value of $\lambda_{\max} = 14.8$ could not be bettered.

The two components of $Q(s)$ are given by

$$Q_1(s) = \frac{0.986(s + 3.26)\,(s + 2.03)\,(s + 0.75)}{(s + 1.68)\,(s^2 + 4.05s + 4.1)} \tag{50a}$$

$$Q_2(s) = \frac{41.92\,(s + 2.04)\,(s^2 + 5.04s + 6.58)}{(s + 1.68)\,(s^2 + 4.05s + 4.1)} \tag{50b}$$

It was reported in Ref. 2 that $Q_2(s)$ is unconstrained and may be taken as zero. To simulate this condition, we set the second columns of the optimal $\tilde{B}$ and $\tilde{D}$ equal to zero. The first positive value of λ for which det$[\phi_{22}(10)]$ changed sign in this case was still observed to be 14.8.

VI. Conclusions

A design methodology for the synthesis of finite-interval H_∞ controllers is presented using state-space methods. Using observer-based controller parameterization, an optimization problem is formulated. A measure of performance for a given controller is defined in terms of the least value of a parameter occurring in a two-point boundary-value problem. Optimality conditions for finding the measure of performance for a given controller are given and the optimization problem seeks to maximize the measure of performance.

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